Chapter 2 Perceptrons

$T \sim \text{Unif}(S(\sqrt{N})^{N-1})$

Vol $\sim \exp\left[\frac{N}{2}[1+\log 2\pi]\right]$

$J \in \mathbb{R}^N \qquad J^2 = N$

$S = \{\pm 1\}^N \Rightarrow S^2 = N$

$R := \frac{J \cdot T}{N} \qquad \varepsilon = \frac{1}{\pi} \arccos R = \frac{\theta}{\pi}$

↑ For Gibbs learning

$\Omega_0(\varepsilon) = \text{Vol}(\text{pts with } R = \cos \pi\varepsilon)$

$\Omega_p(\varepsilon) = \Omega_0(\varepsilon)\,(1-\varepsilon)^p$ ← fraction excluded at each step

$$\Omega_0(\varepsilon) = \int dJ\, \delta(J^2 - N)\, \delta\left(\frac{J \cdot T}{N} - \cos \pi\varepsilon\right)$$

$$\approx \int_{\theta \leq \varepsilon} dJ\, \delta(J^2 - N) = \text{Vol}(\text{pts}) \cdot \sin^N \pi\varepsilon$$

$$\sim \exp\left[\frac{N}{2}\left[1 + \log 2\pi + \log \sin^2 \pi\varepsilon\right]\right]$$

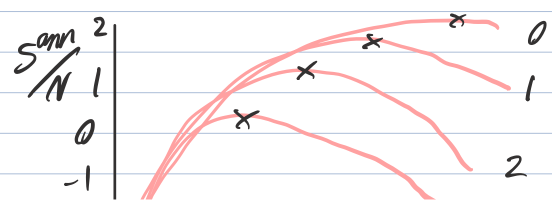
does not concentrate ↙

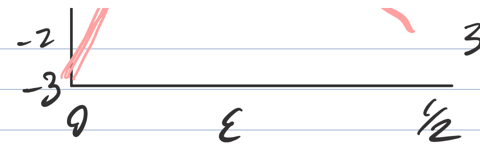
$$\Rightarrow \Omega_p(\varepsilon) \sim \exp\left[N\left(\frac{1}{2}(1+\log 2\pi) + \log \sin \pi\varepsilon + \alpha \log(1-\varepsilon)\right)\right]$$

Saddle point $\Rightarrow \underset{\varepsilon}{\text{argmax}}\ \log \sin \pi\varepsilon + \alpha \log(1-\varepsilon)$

$\Rightarrow \pi \cot(\pi\varepsilon) = \frac{\alpha}{1-\varepsilon} \qquad \alpha \to \infty \Rightarrow \varepsilon \to 0$

$\varepsilon \sim 1/\alpha$





This is wrong : $\Omega(\varepsilon; \{\xi^\mu\}, T)$ is a random var that does not concentrate

$$\log \langle \Omega \rangle \neq \langle \log \Omega \rangle$$

$$S(\varepsilon; \{\xi^\mu\}, T) := \log \Omega$$

↓ new normalization before learning

$$S_0(\varepsilon) \sim \frac{N}{2} \log \sin^2 \pi\varepsilon$$

$$\chi(J; \{\xi^\mu\}, T) := \prod_\mu \theta\left(\frac{T\cdot\xi^\mu \; J\cdot\xi^\mu}{N} \right)$$

$$\Rightarrow \Omega(\xi^\mu, T) = \int d\mu(J)\, \chi(J)$$

$$\Omega_p = \langle\!\langle \Omega(\xi^\mu, T) \rangle\!\rangle_{\xi^\mu, T} \qquad P_T(T) = \mathrm{Unif}(S(\sqrt{N})^{N-1})$$

$$s_{ann} = \log \Omega_p$$

Doing the annealed calculation

$$\lambda_\mu = \frac{1}{\sqrt{N}} J\cdot\xi^\mu \qquad u_\mu = \frac{1}{\sqrt{N}} T\cdot\xi^\mu$$ } Gaussian vars

$$\mathbb{E}\lambda = 0 \quad \mathbb{E} u = 0 \quad \mathbb{E}\lambda u = R \quad \Sigma = \begin{pmatrix} 1 & R \\ R & 1 \end{pmatrix} \Rightarrow \Sigma^{-1} = \frac{1}{1-R^2}\begin{pmatrix} 1 & -R \\ -R & 1 \end{pmatrix}, \sqrt{\det \Sigma} = \frac{1}{1-R^2}$$
$$\mathbb{E}\lambda^2 = 1 \quad \mathbb{E} u^2 = 1$$

$$\Rightarrow \Omega_p = \int_{-1}^{1} dR \int d\mu(J) \prod_\mu d\lambda_\mu du_\mu \prod_\mu \theta(\lambda_\mu u_\mu) \prod_\mu \langle\!\langle \delta(\lambda_\mu - \tfrac{1}{\sqrt{N}} J\cdot\xi^\mu)\, \delta(u_\mu - \tfrac{1}{\sqrt{N}} T\cdot\xi^\mu) \rangle\!\rangle$$

matrix elem of gaussian

$$\Rightarrow \frac{1}{2\pi\sqrt{1-R^2}} \exp\left[-\frac{1}{2(1-R^2)} (\lambda_\mu^2 + u_\mu^2 - 2R\lambda_\mu u_\mu) \right]$$

$$\rightsquigarrow \frac{2}{\sqrt{1-R^2}} \int_0^\infty \frac{d\lambda}{\sqrt{2\pi}} \frac{du}{\sqrt{2\pi}} \exp\left[-\frac{\lambda^2 + u^2 - 2R\lambda u}{2(1-R^2)} \right] = 1 - \frac{1}{\pi} \arccos R$$

$$R = \cos \pi\varepsilon \Rightarrow 1 - R^2 = \sin^2 \pi\varepsilon$$

$$\Rightarrow \Omega_p = \int dR \exp\left[N\left(\tfrac{1}{2} \log(1-R^2) + \alpha \log\left(1 - \tfrac{1}{\pi}\arccos R\right) \right) \right]$$

entropy ↑ energy ↑

$$\Rightarrow s^{ann} = N \cdot \max_R \left\{ \frac{1}{2} \log(1-R^2) + \alpha \log\left(1 - \frac{1}{\pi} \arccos R\right) \right\}$$

$\Rightarrow$ R that contributes most to Ω_p comes from this

$$\Omega^{typ} = \exp \langle\langle \log \Omega \rangle\rangle \neq \langle\langle \Omega \rangle\rangle$$ ← annealed approx

Gardner Analysis:

We want:

$\langle\langle \log \Omega \rangle\rangle \Rightarrow$ use replica theory

$$\langle\langle \Omega^n(\xi, T) \rangle\rangle = \left\langle\left\langle \int \prod_a d\mu(J^a) \prod_\mu \prod_a \theta\left(\frac{T\cdot\xi^\mu \; J^a\cdot\xi^\mu}{N}\right) \right\rangle\right\rangle$$

$$\lambda^a_\mu = J^a \cdot \xi^\mu \qquad u_\mu = T \cdot \xi^\mu$$

$$\langle\langle \lambda^a_\mu \rangle\rangle = \langle\langle u_\mu \rangle\rangle = 0 \qquad \langle\langle \lambda^{\mu 2}_a \rangle\rangle = \langle\langle u_\mu^2 \rangle\rangle = 1$$

$$\langle\langle \lambda^a_\mu \lambda^b_\nu \rangle\rangle = q^{ab} \delta_{\mu\nu} \qquad \langle\langle \lambda^a_\mu u_\nu \rangle\rangle = R^a \delta_{\mu\nu}$$

$$\Rightarrow \langle\langle \Omega^n \rangle\rangle = \int d[J^a, \lambda^a_\mu, u_\mu] \prod_{a,\mu} \theta(\lambda^a_\mu u_\mu) \left\langle\left\langle \delta\left(\lambda^a_\mu - \frac{J^a\xi^\mu}{\sqrt{N}}\right) \delta\left(u_\mu - \frac{T\cdot\xi^\mu}{\sqrt{N}}\right) \right\rangle\right\rangle$$

$$= \int d[\cdots] \prod_{\mu a} \theta(\cdots) \exp\left[i \lambda^a_\mu \hat{\lambda}^a_\mu + i u_\mu \hat{u}_\mu\right] \left\langle\left\langle \exp\left(-\frac{i}{\sqrt{N}} \hat{\lambda}^a_\mu J^a \cdot \xi^\mu - \frac{i}{\sqrt{N}} \hat{u}_\mu T \cdot \xi^\mu\right) \right\rangle\right\rangle$$

independence of J, ξ^μ & their components →

$$\left\langle\left\langle \prod_{i,\mu} \left\langle\left\langle \exp\left[-\frac{i}{\sqrt{N}} \left(\sum_a \hat{\lambda}^a_\mu J^a_i + u_\mu T\right) \xi^\mu_i \right] \right\rangle\right\rangle_\xi \right\rangle\right\rangle_T$$

($\xi^\mu_i = \pm 1$)

$$= \left\langle\left\langle \prod_{i,\mu} 2\cos\left[\frac{1}{\sqrt{N}} \sum_a \left(\hat{\lambda}^a_\mu J^a_i + u_\mu T\right)\right] \right\rangle\right\rangle_T$$

fun fact: notice it's the same as if $\xi \sim \mathcal{N}(0, \mathbb{I})$

2 is irrel

$$= \left\langle\left\langle \exp\left[\sum_{i,\mu} \log \cos \frac{1}{\sqrt{N}} \sum_a (\cdots)\right] \right\rangle\right\rangle_T$$

$T \cdot T = N$ ↓

large N (this expansion is only justified post facto, observing $\hat{\lambda} J + u \sim \mathcal{O}(\sqrt{N})$ contributions are supressed)

$$= \left\langle\left\langle \exp\left[-\frac{1}{2N} \sum_{i,\mu} \left(\hat{\lambda}^a_\mu \hat{\lambda}^b_\mu J^a_i J^b_i + 2 \hat{\lambda}^a_\mu J^a_i u_\mu T_i + u_\mu^2 T_i^2\right)\right] \right\rangle\right\rangle_T$$

einsum on a,b

$$\Rightarrow \zeta = i \sum_{a,\mu} \hat{\lambda}^a_\mu \lambda^a_\mu + i \sum_\mu \hat{u}_\mu \cdot u_\mu - \frac{1}{2N} \sum_{ab} \hat{\lambda}^a_\mu \hat{\lambda}^b_\mu \underbrace{J^a \cdot J^b}_{N q^{ab}} - \frac{1}{N} \sum_{a,\mu} \hat{\lambda}^a_\mu u_\mu \underbrace{J^a \cdot T}_{R^a} - \frac{1}{2} \sum u_\mu^2$$

$$\Omega^n = \int d[\cdots] \prod_{a,\mu} \theta[\cdots] \langle\langle \exp \tilde{S} \rangle\rangle_T$$

Take $q^{ab} = \frac{1}{N} J^a \cdot J^b$ $\quad R^a = \frac{1}{N} T \cdot J^a$

$$\int \prod_{a<b} N dq^{ab} \prod_a N dR^a \, d\mu(J^a) \langle\langle \prod_a \delta(J^a \cdot T^a - NR^a) \rangle\rangle_T \prod_{a<b} \delta(J^a \cdot J^b - N q^{ab})$$

$$\times \int d\lambda \frac{d\hat{\lambda}}{2\pi} du \frac{d\hat{u}}{2\pi} \prod_{a,\mu} \theta(u_\mu \lambda^a_\mu) \exp\left[i \lambda \cdot \hat{\lambda} + i u \cdot \hat{u} - \frac{1}{2} \hat{\lambda}^a_\mu \hat{\lambda}^b_\mu q^{ab} - \hat{\lambda}^a_\mu \hat{u}_\mu R^a - \frac{1}{2} \hat{u}_\mu^2 \right]$$

average redundant by isotropy of $\{\xi^\mu\}$ $\Rightarrow$ isotropy of J^a

$\hat{q}^{ab}$ for $\frac{1}{N} J^a \cdot J^b = q^{ab}$ $\quad q^{aa} = 1$

Do gaussian $\hat{u}_\mu$ integral $\frac{1}{2}(-iu_\mu + \hat{\lambda}^a_\mu R^a)^2 = -\frac{1}{2} u \cdot u - i u_\mu \hat{\lambda}^a_\mu R^a + \frac{1}{2} \hat{\lambda}^a_\mu \hat{\lambda}^b_\mu R^a R^b$

$$\Omega^n = \int d\left[\frac{q^{ab} \hat{q}^{ab}}{2\pi/N} \frac{R^a \hat{R}^a}{2\pi/N} \frac{J^a_i}{\sqrt{2\pi e}} \frac{u_\mu}{\sqrt{2\pi}} \lambda^a_\mu \frac{\hat{\lambda}^a_\mu}{2\pi} \right] \exp\left(\frac{iN}{2} q^{ab} \hat{q}^{ab} + iN R^a \hat{R}^a \right)$$

1) factors over i: $\times \exp\left(-\frac{i}{2} \hat{q}^{ab} J^a_i J^b_i - i \sum_a \hat{R}_a J^a_i \right)$

2) factors over μ: $\times \prod_{a,\mu} \theta(\lambda^a_\mu u_\mu) \exp\left[-\frac{1}{2} \hat{\lambda}^a_\mu \hat{\lambda}^b_\mu (q^{ab} - R^a R^b) + i \lambda^a_\mu \hat{\lambda}^a_\mu - i u_\mu \hat{\lambda}^a_\mu R^a - u_\mu^2/2 \right]$

$$S_{eff} = \frac{i}{2} q^{ab} \hat{q}^{ab} + i R^a \hat{R}^a + G_S + \alpha G_E$$

1): $\exp\left[N \cdot G_S(k, \hat{q}^{ab}, \hat{R}^a) \right]$

$$G_S = \log \int \prod_a \frac{dJ^a}{\sqrt{2\pi e}} \exp\left(-\frac{i}{2} J^a J^b \hat{q}^{ab} - i \hat{R}^a J^a \right)$$

(1-d int)

$\hat{Q}_{ab} := \hat{q}^{aa} \delta_{ab} + i \hat{q}^{ab} (1 - \delta_{ab})$ (diag; off-diag)

$$= -\frac{n}{2} \log 2\pi e + \frac{n}{2} \log 2\pi - \frac{1}{2} \log \det \hat{Q} - \frac{1}{2} R \cdot \hat{Q}^{-1} \cdot Q$$

$$= -\frac{n}{2} - \frac{1}{2} \operatorname{Tr} \log \hat{Q} - \frac{1}{2} R \cdot \hat{Q}^{-1} R$$

2) $\exp[N\alpha\, G_E(q^{ab}, R^b)]$

$$G_E = \log \int \frac{du}{\sqrt{2\pi}} \prod_a \frac{d\lambda^a d\hat{\lambda}^a}{2\pi} \Theta(u\lambda^a) \exp\left(-\frac{u^2}{2} - \frac{1}{2}\hat{\lambda}^a\hat{\lambda}^b(q^{ab} - R^a R^b) + i\lambda^a\hat{\lambda}^a - iu\hat{\lambda}^a R^a\right)$$

1d ints

$Dt = dt\, \frac{e^{-t^2/2}}{\sqrt{2\pi}}$

$Du = du\, \frac{e^{-u^2/2}}{\sqrt{2\pi}}$

$$= \log 2 \int \frac{du}{\sqrt{2\pi}} \int_0^\infty [d\lambda^a du] \int \frac{d\hat{\lambda}}{2\pi} \exp(\cdots)$$

Hubbard on $\hat{\lambda}$ — RS ansatz

$$= \log 2 \int Dt \int_0^\infty Du \int_0^\infty \prod_a d\lambda^a \int \frac{d\hat{\lambda}_a}{2\pi} \exp\left[-\frac{1-q}{2}(\hat{\lambda}^a)^2 + i\,\hat{\lambda}^a(\lambda^a - uR - \sqrt{q-R^2}\,t)\right]$$

$$= \log 2 \int Dt \int_0^\infty Du \left[\int_0^\infty \frac{d\lambda}{\sqrt{2\pi(1-q)}} \exp\left[-\frac{1}{2(1-q)}(\lambda - uR - \sqrt{q-R^2}\,t)^2\right]\right]^n$$

$$= \log 2 \int Dt \int_0^\infty Du\, H^n\left(-\frac{\sqrt{q-R^2}\,t + Ru}{\sqrt{1-q}}\right)$$

$t \to \frac{\sqrt{q}\,t - Ru}{\sqrt{q-R^2}}$ $\quad \sqrt{q-R^2}\,t + Ru \to \sqrt{q}\,t$

$t^2 \to \frac{q}{q-R^2}\frac{t^2}{2} + \frac{1}{2}\frac{R^2u^2}{q-R^2} + \frac{Ru\sqrt{q}\,t}{q^2-R^2}$

$$= \log 2 \int Dt\, H^n\left(-\frac{\sqrt{q}}{\sqrt{1-q}}t\right) \int_0^\infty du \sqrt{\frac{q}{q-R^2}} \exp\left[-\frac{u^2}{2} - \frac{1}{2}\frac{R^2}{q-R^2}u^2 - \frac{Ru\sqrt{q}\,t}{q-R^2} + \frac{R^2t^2}{q-R^2}\right]$$

$= \frac{t^2}{2} + \frac{R^2}{q-R^2}\frac{t^2}{2} + \frac{1}{2}\frac{R^2u^2}{q-R^2} + \frac{Ru\sqrt{q}\,t}{q-R^2}$

$$= \log 2 \int Dt\, H^n(\cdots) \int_0^\infty du\, \exp\left[-\frac{(\sqrt{q}\,u + Rt)^2}{q-R^2}\right] \sqrt{\frac{q}{q-R^2}}$$

genius step...

$$= \log 2 \int Dt\, H^n\left(-\sqrt{\frac{q}{1-q}}\,t\right) H\left(\frac{-Rt}{\sqrt{q^2-R^2}}\right)$$

← note @ $n=0$ this $= \log 2 \frac{1}{2} = 0$

Back to 1) $\quad N \to \infty \Rightarrow$ Saddle point. $Q_{ab} := \delta_{ab} + q^{ab}(1-\delta_{ab})$

$$S_{eff} = -\frac{n}{2} - \frac{1}{2}\mathrm{Tr}\log\hat{Q} - \frac{1}{2}\hat{R}\hat{Q}^{-1}\hat{R} + \frac{1}{2}\mathrm{Tr}\, Q\hat{Q} + i\, R\cdot\hat{R} \quad + (\hat{R}, \hat{Q} \text{ indep})$$

$$0 = \frac{\partial S_{eff}}{\partial \hat{R}} = -\hat{Q}^{-1}\hat{R} + iR$$

$$0 = \frac{\partial S_{eff}}{\partial \hat{Q}} = -\frac{1}{2}\hat{Q}^{-1} + \frac{1}{2}(\hat{R}\cdot\hat{Q}^{-1})(\hat{R}\cdot\hat{Q}^{-1})^T + \frac{1}{2}Q$$

$\frac{\partial S}{\partial Q, R}$ is harder

$\Rightarrow \hat{R} = i\hat{Q}\cdot R$

$\hat{Q}^{-1} = Q - RR^T =: C$

$\Rightarrow Q\hat{Q} = \mathbb{1} + \hat{Q}RR^T$

$\Rightarrow \frac{1}{2}\mathrm{Tr}\, Q\hat{Q} = \frac{n}{2} + \frac{1}{2}R\cdot\hat{Q}\cdot R = \frac{n}{2} - \frac{i}{2}R\cdot\hat{R}$

$\Rightarrow \frac{1}{2}\mathrm{Tr}\, Q\hat{Q} + iR\hat{R} = \frac{n}{2} + \frac{1}{2}\hat{R}\cdot\hat{Q}^{-1}\cdot\hat{R}$ ★★★

$$\Rightarrow N S_{eff}(Q, R) = N\left[\frac{1}{2}\mathrm{Tr}\log C + G_E(Q, R)\right]$$

specific values q^{ab}, R^a

dominate:

Replica symmetry : $q^{ab} = q \qquad R^a = R$

$$\Rightarrow \frac{1}{2} Tr \log C = \frac{1}{2} Tr \log \left[\begin{pmatrix} 1 & q \\ q & 1 & \\ & & \ddots \end{pmatrix} - R 1 1^T \right]$$

1 eig $= (1-q) + n(q-R^2)$

$n-1$ eigs $= (1-q)$

$$\Rightarrow \frac{1}{2}(n-1)\log(1-q) + \frac{1}{2}\log[(1-q)+n(q-R^2)]$$

$$= \frac{n}{2}\log(1-q) + \frac{1}{2}\log\left(1 + n\frac{q-R^2}{1-q}\right)$$

$$\to \frac{n}{2}\log(1-q) + \frac{n}{2}\frac{q-R^2}{1-q}$$

$$\Rightarrow \langle \Omega^n \rangle = \exp\left[Nn\left[\frac{1}{2}\log(1-q) + \frac{1}{2}\frac{q-R^2}{1-q} + 2\alpha \int Dt\, H\left(-\frac{Rt}{\sqrt{q-R^2}}\right)\log H\left(-\sqrt{\frac{q}{1-q}}t\right)\right]\right]$$

$$\Rightarrow \frac{1}{N}\langle \log \Omega \rangle = \underset{q,R}{\text{extr}}\left[\frac{1}{2}\log(1-q) + \frac{1}{2}\frac{q-R^2}{1-q} + 2\alpha\int Dt \;\cdots\right]$$

$$\frac{\partial}{\partial q} = 0 \Rightarrow \frac{q-R^2}{2(1-q)^2} = \frac{\alpha}{\sqrt{2\pi}}\int Dt\, \frac{t}{2}\left[\frac{1}{\sqrt{q}(1-q)^{3/2}}\frac{H(\frac{-Rt}{\sqrt{q-R^2}})\exp[-\frac{qt^2}{2(1-q)}]}{H(-\frac{\sqrt{q}}{\sqrt{1-q}}t)} - \frac{R\log H(\cdots t)\exp[-\frac{R^2t^2}{q-R^2}]}{(q-R^2)^{3/2}}\right]$$

$$= \frac{\alpha}{2\pi}\frac{1-q}{\sqrt{q}(1-q)^{3/2}}\frac{\sqrt{q}}{\sqrt{1-q}}\frac{1}{2}\int Dt\, \frac{H(-Rt\cdots)\exp[-\frac{qt^2}{1-q}]}{H^2(-\frac{\sqrt{q}}{\sqrt{1-q}}t)} + \frac{\alpha}{2\pi}\frac{1}{2}\int Dt\, \frac{\exp[-R\cdots - q\cdots]}{H(\cdots\sqrt{q})}\left[\frac{R(1-q)}{\sqrt{q}(1-q)^{3/2}\sqrt{q-R^2}} - \frac{R\sqrt{q}(q-R^2)}{q\sqrt{1-q}(q-R^2)^{3/2}}\right] \to 0$$

$$* \Rightarrow \frac{q-R^2}{1-q} = \frac{\alpha}{\pi}\int Dt\, H\left(-\frac{Rt}{\sqrt{q-R^2}}\right)\frac{\exp[-\frac{qt^2}{1-q}]}{H^2(-\sqrt{\frac{q}{1-q}}t)}$$

$$\frac{\partial}{\partial R} = 0 \Rightarrow \frac{R}{1-q} = \frac{2\alpha\, q}{(q-R^2)^{3/2}}\int Dt\, t\, \log H\left[-\frac{\sqrt{q}}{\sqrt{1-q}}t\right] H'\left[\frac{-Rt}{\sqrt{q-R^2}}\right]$$

$$\frac{R(q-R^2)^{3/2}}{q(1-q)} = \frac{2\alpha}{\sqrt{2\pi}}\int Dt\, t \log H(\cdots)\exp\left[-\frac{t^2}{2}\frac{R^2}{q-R^2}\right]$$

$$= \frac{2\alpha}{2\pi}\sqrt{\frac{q}{1-q}}\frac{q-R^2}{q}\int Dt\, \frac{\exp[-\frac{t^2}{2}[\frac{R^2}{q-R^2}+\frac{q}{1-q}]]}{H(-\frac{\sqrt{q}}{\sqrt{1-q}}t)}$$

$$* \Rightarrow \frac{R\sqrt{q-R^2}}{\sqrt{q}\sqrt{1-q}} = \frac{\alpha}{\pi}\int Dt\, \frac{\exp[-\frac{t^2}{2}[\frac{R^2}{q-R^2}+\frac{q}{1-q}]]}{H(-\sqrt{q}\cdots)}$$

$H\left(-\frac{\sqrt{q}}{\sqrt{1-q}}t\right)$

$q=R \Rightarrow$ Eqs coincide "Typical Student-Student = typical Student-Teacher"

$$\Rightarrow q = R = \frac{\alpha}{\pi}\int Dt \; \frac{\exp\left[-\frac{Rt^2}{1-R}\right]}{H\left[-\frac{\sqrt{R}}{\sqrt{1-R}}t\right]}$$

$\frac{t}{\sqrt{1-R}} \to t$

$\frac{t^2}{2} \to \frac{t^2}{2} - \frac{Rt^2}{2}$

$dt \to \sqrt{1-R}\,dt$

$$= \frac{\alpha}{\pi}\sqrt{1-R}\int Dt \; \frac{\exp[-Rt^2]}{H(-\sqrt{R}t)}$$

$$\Rightarrow \frac{1}{N}\langle\langle \log \Omega(\xi^\mu, T)\rangle\rangle_{\xi, T} = \underset{R}{\text{argmax}}\left[\frac{1}{2}\log(1-R) + \frac{R}{2} + 2\alpha\int Dt\, H\left(-\frac{\sqrt{R}}{\sqrt{1-R}}t\right)\log H\left(-\frac{\sqrt{R}}{\sqrt{1-R}}t\right)\right]$$

Back to Chapter 2

$$\langle\langle Z^n\rangle\rangle = \int d\mu(J^a)\left\langle\left\langle \theta\left(\frac{T\cdot\xi^\mu}{\sqrt{N}}\frac{1}{\sqrt{N}}J^a\xi^\mu\right)\right\rangle\right\rangle$$

$\begin{matrix}\lambda^a & 0\\ 0 & u\end{matrix} \; : \; \begin{pmatrix}1 & R \\ R & 1\end{pmatrix} = (1-R)\mathbb{1}_{n\times n} + R\,\mathbb{1}\mathbb{1}^T$

$\lambda^a_\mu \sim N(0, \mathbb{1}_\mu \otimes Q_{ab})$

$u_\mu \sim N(0,1)$

$$= \int d\mu(J^a)\, d\lambda^a_\mu\, du_\mu \left\langle\left\langle \delta\left(\lambda^a_\mu - \frac{1}{\sqrt{N}}J^a\cdot\xi^\mu\right)\delta\left(u_\mu - \frac{1}{\sqrt{N}}T\cdot\xi^\mu\right)\right\rangle\right\rangle_{T\xi}\theta(\lambda^\mu_a u_\mu)$$

$\langle\langle \lambda^a_\mu u_\mu\rangle\rangle = \frac{J^a\cdot T}{N} = R$

$$= \int d\mu[(J^a), \lambda, u]\,\theta(\lambda^\mu_a u_\mu)\left\langle\left\langle\delta\left(\frac{J^a\cdot T}{N} - R\right)\right\rangle\right\rangle_T \delta\left(\frac{J^a\cdot J^b}{N} - Q_{ab}\right)\prod_\mu \exp[\ldots]\frac{1}{\sqrt{\ldots}}$$

$\begin{pmatrix}1+(n-1)R & -R \\ -R & \ddots\end{pmatrix}$

eigs: $(1-R) + (n-1)R$, $1-R \Rightarrow (1-R)^n(1+nR)$

$$((1-R)\mathbb{1} + R\,\mathbb{1}\mathbb{1}^T)^{-1} = \frac{1}{1-R}\left[\mathbb{1} - \frac{R}{1-R}\frac{\mathbb{1}\mathbb{1}^T}{1+\frac{(n-1)R}{1-R}}\right] = \frac{\mathbb{1}}{1-R} - \frac{R\,\mathbb{1}\mathbb{1}^T}{(1-R)(1+nR)} = \frac{1}{(1-R)(1+nR)}\left[\mathbb{1}(1+nR) - R\,\mathbb{1}\mathbb{1}^T\right]$$

$$= \int dJ^a \frac{d\lambda^a_\mu\, d\lambda^0_\mu}{\sqrt{2\pi(1-R)}\,2\pi(1+nR)}\,\theta(\cdot\cdot)\left\langle\left\langle\delta\left(\frac{J^a\cdot T}{N} - R\right)\right\rangle\right\rangle\delta\left(\frac{J^a\cdot J^b}{N} - R\right)\prod_\mu \exp\left[-\frac{1}{2}\frac{1}{(1-R)(1+nR)}\left((1+(n-1)R)\sum_a(\lambda^a_\mu)^2 - \frac{R}{2}\sum_{a\neq b}\lambda^a_\mu\lambda^b_\mu\right)\right]$$

(u_μ above λ^0_μ; $a\in\{0,\ldots,n\}$)

$$= \int_{-1}^{1} dR \int d\mu(J^a)\,\delta\left(\frac{J^a\cdot J^b}{N} - R\right)\left\{\frac{2\sqrt{1-R}}{\sqrt{1+nR}}\int_0^\infty \frac{d\lambda_a}{\sqrt{2\pi(1-R)}}\exp\left[-\frac{1}{2(1-R)(1+nR)}\left((1+(n-1)R)(\lambda^a)^2 - \frac{R}{2}\lambda^a\lambda^b\right)\right]\right\}^{\alpha N}$$

↑ does not factorize in a

1) $\int_{-1}^{1} dR \frac{dJ^a}{\sqrt{2\pi e}}\frac{d\hat{Q}_{ab}}{2\pi/N}\exp\left[\frac{i}{2}\hat{Q}_{ab}(J^a\cdot J^b - NQ_{ab})\right]$ $\quad Q_{ab} = \begin{pmatrix}1 & R \\ R & \ddots\end{pmatrix}$

$$\rightarrow \int \frac{dJ^a\,\hat{Q}_{ab}}{2\pi/N}\exp\left[\frac{iN}{2}\hat{Q}_{ab}Q_{ab} - \frac{N}{2}\mathrm{Tr}\log\hat{Q}\right]$$

$\frac{\delta}{\delta\hat{Q}} = 0 \Rightarrow iQ_{ab} = [\hat{Q}^{-1}]_{ab}$ $\qquad \frac{1}{2}\mathrm{Tr}\log\left(\mathbb{1}(1-R) + \mathbb{1}\mathbb{1}^T R\right)$

$$\Rightarrow N G_S = N\left(\frac{n}{2}\log(1-R) + \frac{1}{2}\log(1+nR)\right)$$

2)

$$= \frac{2\sqrt{1-R}}{\sqrt{1+nR}}\int_0^\infty \frac{d\lambda_a}{\sqrt{2\pi(1-R)}} \exp\left[-\frac{1}{2(1-R)(1+nR)}\left((1+nR)(\lambda^a)^2 - R\,\lambda^a\lambda^b\right)\right]$$

$$= \frac{2\sqrt{1-R}}{\sqrt{1+nR}}\int Dt \int_0^\infty \frac{d\lambda_a}{\sqrt{2\pi(1-R)}} \exp\left[-\frac{1}{2}\frac{\lambda^2}{1-R} + \frac{\sqrt{R}}{\sqrt{1-R}\sqrt{1+nR}}\lambda t\right]$$

$$= 2\sqrt{\frac{1-R}{1+nR}}\int Dt \int_0^\infty \frac{d\lambda}{\sqrt{2\pi(1-R)}} \exp\left[-\frac{1}{2(1-R)}\left[\left(\lambda - \sqrt{R}\frac{\sqrt{1-R}}{\sqrt{1+nR}}t\right)^2 - \frac{R(1-R)}{(1+nR)}t^2\right]\right]$$

$$t' = \sqrt{\frac{1-R}{1+nR}}\,t$$

$$= 2\int dt \left(\int_0^\infty \frac{d\lambda}{\sqrt{2\pi(1-R)}} \exp\left[-\frac{1}{2(1-R)}\left[\left(\lambda - \frac{\sqrt{R}}{2}t\right)^2 - Rt^2\right]\right]\right)^n \exp\left(-\frac{1+nR}{1-R}\frac{t^2}{2}\right)$$

$$= 2\int dt \; H^{n+1}\left(-\frac{\sqrt{R}}{\sqrt{1-R}}t\right)\exp\left(-\frac{(1+nR)-nR}{1-R}\;\frac{t^2}{2}\right)$$

$$= 2\int Dt\, H^{n+1}\left(-\sqrt{\frac{R}{1-R}}\,t\right) \quad ✓$$

$$\Rightarrow G_E = \log 2\int Dt\, H^{n+1}\left(-\sqrt{\frac{R}{1-R}}\,t\right)$$

$$\Rightarrow N S_{eff} = N G_S + \alpha N G_E$$
$$= N\left[\frac{n}{2}\log(1-R) + \frac{1}{2}\log(1+nR) + \alpha\log 2\int Dt\, H^{n+1}(\dots)\right]$$

$$\Rightarrow \langle\langle \log\Omega\rangle\rangle = N\max_R\left[\frac{1}{2}\log(1-R) + \frac{R}{2} + 2\alpha\int Dt\, H\left(-\sqrt{\frac{R}{1-R}}t\right)\log H\left(-\sqrt{\frac{R}{1-R}}t\right)\right]$$

$$\frac{\partial}{\partial R} \Rightarrow \frac{R}{\sqrt{1-R}} = \frac{\alpha}{\pi}\int Dt\,\frac{e^{-Rt^2/2}}{H(-\sqrt{R}t)}$$

as before!

Chapter 3: Learning Rules

3.1 Hebb

ξ_i^μ input i for "channel" J_i of μth example

$\sigma_T^\mu = \mathrm{sgn}(T\cdot\xi^\mu)$ required output

old style IF $(\sigma_T^\mu)_i$ ξ_i $J_i \Rightarrow J_i + 1$ else none

symmetric IF $(\sigma_T^\mu)_i$ ξ_i $J_i \Rightarrow J_i + 1$ else $J_i \to J_i - 1$

$$J \to J + \xi^\mu \sigma_T^\mu$$

$$\Rightarrow J^H = \frac{1}{\sqrt{N}} \sum_\mu \xi^\mu \sigma_T^\mu$$

$$\Delta^\nu := \frac{1}{\sqrt{N}} J\cdot\xi^\nu \sigma_T^\nu$$

$\Delta > 0 \Rightarrow$ correct

$$\Delta^\nu = \frac{1}{N} \sum_\mu \xi^\mu\cdot\xi^\nu \sigma^\mu \sigma^\nu$$

$$= 1 + \frac{1}{N} \sum_{\mu\neq\nu} \xi^\mu\cdot\xi^\nu \sigma_T^\mu \sigma_T^\nu$$

signal (the 1); noise, ie crosstalk (the sum)

$\xi_{\parallel}^\mu \sim N(0,1)$ $\quad \frac{e^{-x^2/2}}{\sqrt{2\pi}}$

$|\xi_\parallel| \sim \sqrt{\frac{2}{\pi}} e^{-x^2/2}$ ($N_{>0}$) $\Rightarrow E[|\xi_\parallel|] = \sqrt{\frac{2}{\pi}}$

$\mathrm{Var} \sim N$

Take $\xi^\mu \sim N(0, \sqrt{N}\,\mathbb{1})$

$$\xi^\mu = \xi_\parallel^\mu T + \xi_\perp^\mu$$

$$\Delta^\nu = 1 + \frac{\xi_\parallel^\nu \sigma_T^\nu}{N} \sum_{\mu\neq\nu} \xi_\parallel^\mu \sigma_T^\mu + \frac{\xi_\perp^\nu \sigma_T^\nu}{N} \cdot \sum_{\mu\neq\nu} \xi_\perp^\mu \sigma_T^\mu$$

$\xi_\parallel^\mu \sigma_T^\nu = |\xi_\parallel^\mu|$

sum of $p-1$ iid vars >0 $\to N\alpha\sqrt{\frac{2}{\pi}}|x|$

on $N-1$ sphere; sum of $N(p-1)$ terms $\to N\sqrt{\alpha}\, y$ $\quad y \sim N(0,1)$

$|x| \sim \mathcal{N}_{\geq 0}$

$$P(\xi^\mu \text{ misclassified}) = \mathbb{P}\left(1 + \alpha\sqrt{\tfrac{2}{\pi}}|x| + \sqrt{\alpha}\, y < 0\right)$$

$$\Rightarrow \varepsilon_{train} = 2\int_0^\infty Dx \int_{-\infty}^{-\frac{1}{\sqrt{\alpha}} - \sqrt{\frac{2\alpha}{\pi}}x} Dy = 2\int_0^\infty Dx\, H\left(\frac{1}{\sqrt{\alpha}} + \sqrt{\frac{2\alpha}{\pi}}\, x\right)$$

Generalization:

$$\frac{J^H \cdot T}{N} = \frac{1}{N}\sum_\mu \frac{T\cdot\xi^\mu \sigma^\mu}{\sqrt{N}} = \underbrace{\frac{1}{N}\sum_\mu |\xi_\parallel^\mu|}_{\alpha\sqrt{\frac{2}{\pi}}}$$

$$J_H \cdot J_H = \frac{1}{N}\sum_{\mu,\nu} \xi^\mu \xi^\nu \sigma_T^\mu \sigma_T^\nu$$

$$= \sum_\mu \frac{\xi^\mu \cdot \xi^\mu}{N} \cancel{\sigma_T^\mu \sigma_T^\mu} + \underbrace{\sum_{\mu\neq\nu} \frac{\xi_\parallel^\mu \xi_\parallel^\nu}{N} \sigma_T^\mu \sigma_T^\nu}_{} + \underbrace{\sum_{\mu\neq\nu} \xi_\perp^\mu \xi_\perp^\nu \sigma_T^\mu \sigma_T^\nu}_{0}$$

$$= \alpha N \qquad \underbrace{\frac{1}{N}\sum_\mu |\xi^\mu| \sum_\nu |\xi_\parallel^\nu|}_{N\alpha^2 \frac{2}{\pi}}$$

$$= \alpha N(1 + \alpha 2/\pi)$$

$$\Rightarrow |J| = \sqrt{\alpha N(1 + 2\alpha/\pi)} \qquad |T| = N$$

$$J\cdot T = N\alpha\sqrt{\tfrac{2}{\pi}}$$

$$\Rightarrow \varepsilon_g = \frac{1}{\pi}\arccos\frac{J\cdot T}{|J||T|} = \frac{1}{\pi}\arccos\sqrt{\frac{2\alpha}{\pi + 2\alpha}}$$

$$\Rightarrow \varepsilon_t(\alpha) \sim \sqrt{\tfrac{2\alpha}{\pi}}\exp\left(-\tfrac{1}{2\alpha} + \tfrac{1}{\pi}\right) H\left(\sqrt{\tfrac{2}{\pi}}\right)$$

$$\varepsilon_g(\alpha) \sim \frac{1}{2} - \frac{\sqrt{2\alpha}}{\pi^{3/2}}$$

} $\alpha \to 0$

$$\varepsilon_t \sim \varepsilon_g \sim \frac{1}{\sqrt{2\pi\alpha}} \qquad \alpha \to \infty$$

3.2 Perceptron Rule

$J^H = \frac{1}{\sqrt{N}} \sum x^\mu \xi^\mu \sigma_T^\mu$ all x^μ equal

this "leveling of information" causes the poor performance

→ Omit updates for correctly classified examples

$$J \to \begin{cases} J + \frac{1}{\sqrt{N}} \xi^\mu \sigma_T^\mu & J \xi^\mu \sigma_T^\mu < 0 \\ J & \text{else} \end{cases}$$ ⇐ Perceptron

Proof of convergence

X updates: $J_X = \frac{1}{\sqrt{N}} \sum_\mu x^\mu \xi^\mu \sigma_T^\mu$ x^μ = # updates for example μ

$\sum x^\mu = X$

by assumption, $\exists\, J^*$ that interpolates:

$$\frac{1}{\sqrt{N}} J^* \xi^\mu \sigma^\mu \geq \kappa > 0$$ WLOG $|J^*| = N$

← stability

$$\Rightarrow (X\kappa)^2 \leq \left(\frac{1}{\sqrt{N}} J^* \xi^\mu \sigma^\mu x^\mu\right)^2$$

$$= (J^* \cdot J_X)^2 \leq |J^*|^2 |J_X|^2 = N |J_X|^2$$

Say ξ^ν had the last update

$$J_{X-1} \xi^\nu \sigma_T^\nu < 0 < \kappa\sqrt{N}$$

$$\Rightarrow |J_X|^2 = |J_{X-1} + \frac{1}{\sqrt{N}} \xi^\mu \sigma^\nu|^2 = |J_{X-1}|^2 + \frac{2}{\sqrt{N}} J_{X-1} \xi^\nu \sigma_T^\nu + \frac{1}{N} |\xi^\nu|^2$$

$$\leq |J_{X-1}|^2 + 2\kappa + 1$$

⇒ maximum increase in $|J_X|^2$ as X steps is $2\kappa + 1$

$$\Rightarrow X^2 \kappa^2 \leq N X (2\kappa + 1)$$

$$\Rightarrow X \leq N\left(\frac{2}{\kappa} + \frac{1}{\kappa^2}\right) \Rightarrow \text{finite!}$$

3.3 PseudoInverse